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A Multi-Length-Scale Theory of the Anomalous Mixing-Length Growth for Tracer Flow in Heterogeneous Porous Media

Qiang Zhang[1]

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We develop a multi-length-scale (multifractal) theory for the effect of rock heterogeneity on the growth of the mixing layer of the flow of a passive tracer through porous media. The multifractal exponent of the size of the mixing layer is determined analytically from the statistical properties of a random velocity (permeability) field. The anomalous diffusion of the mixing layer can occur both on finite and on asymptotic length scales.

KEY WORDS: Random field; porous media; multifractals; heterogeneity; anomalous diffusion.

1. INTRODUCTION

Anomalous (non-Fickian) diffusive mixing induced by a random velocity field has considerable interest both for oil recovery processes and groundwater ecology. Multi-length-scale rock heterogeneity is instrumental in the generation of anomalous diffusion; it is a primary factor limiting total oil recovery in enhanced oil recovery processes,[23] and it leads to a rapid growth rate for contaminant plumes. Heterogeneity-induced diffusive mixing has been studied from a number of points of view, including multi-scale asymptotic expansions,[30] hierarchical random field models,[15,26] homogenization, weak limits and compensated compactness,[1,29] the renormalization group method,[2,8,16] and Monte Carlo simulation.[9,28] For other treatments of diffusion induced by random fields see, e.g., refs. 3–7, 13, 14, 17–25, and 27). The present work goes beyond previous studies in

[1] Department of Applied Mathematics and Statistics, SUNY at Stony Brook, Stony Brook, New York 11794-3600.

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its analysis of multiple-length-scale anomalous (non-Fickian) and transient diffusive behavior.

Glimm and Sharp[15] established a random field model to study the effect of rock heterogeneity on the exponent of the mixing length. This model predicts that for a fractal permeability field, i.e., a permeability field whose covariance function obeys a pure power law, the diffusive mixing layer can be anomalous when the permeability field correlation decays slowly. Numerical simulations of tracer flow in two dimensions have confirmed their prediction.[10–12] The main result of this paper is a multi-length-scale theory which generalizes the results of ref. (15) to a multi-fractal, anisotropic random velocity field. We find: the mixing exponent at length scale l is expressed as an average over all length scales smaller than l, weighted by the velocity field correlation function. In case the exponent of the velocity correlation is slowly varying as a function of length scale, it determines the mixing length exponent on that length scale by a simple formula. Our results also show that normal diffusion can be achieved asymptotically if and only if the diffusion coefficient converges.

Let $f(l)$ be a positive statistical quantity measured at a length scale l from a random field. The exponent of f is defined as

$$\text{exponent of } f(l) \equiv \frac{d \ln f(l)}{d \ln l} \tag{1.1}$$

A fractal field is the special case in which $d \ln f(l)/d \ln l = \text{const}$. Otherwise it is multifractal. A velocity field, determined from a random permeability field, is also random and has length scale dependence. When two fluid phases flow through such a random velocity field, a mixing layer is developed at the interface between the phases. The mixing length, defined as the size of the mixing layer in the direction of the flow, increases as time evolves. We determine the growth of the mixing layer from the rock heterogeneity. Both experimental and field data show a striking phenomenon, known as "the scale effect": the dispersivity increases systematically (but perhaps not as a pure power law) with length scale.[13,24] Our theory explains this phenomenon.

Multi-length-scale rock heterogeneity, microscopic (molecular) diffusion, and fluid instabilities all contribute to the mixing process. In order to study the effects due to heterogeneity alone, we consider unit mobility miscible displacement (tracer or tagged flow). Tracer flow is a common case for groundwater ecology. We consider incompressible flow in d dimensions. The motion of the fluids is governed by a linear transport equation,

$$s_t + \mathbf{v} \cdot \nabla s = 0 \tag{1.2}$$

where s is the saturation value of the fluid, $s=1$ for tagged fluid and $s=0$ for untagged fluid. ∇ is the spatial gradient operator. We assume the velocity field $\mathbf{v}$ is stationary, random, and Gaussian. Therefore it is a function of the spatial variables only and the statistical behavior of the field is determined uniquely by the mean value and the two-point covariance function of the field. More fundamentally, the velocity field $\mathbf{v}$ is determined by the random permeability tensor K from Darcy's law and the condition of incompressibility

$$\mathbf{v} = -\frac{K}{\mu}\nabla P \qquad \text{and} \qquad \nabla\cdot\mathbf{v}=0 \tag{1.3}$$

Here μ is the fluid viscosity and P is the pressure.

Let $\langle s(t,\mathbf{x})\rangle$ be the saturation value after ensemble average of the random velocity $\mathbf{v}$. Applying a systematic expansion of the solution in terms of the fluctuation of the random velocity field, one can show that $\langle s(t,\mathbf{x})\rangle$ satisfies the convection-diffusion equation,

$$\frac{\partial\langle s(t,\mathbf{x})\rangle}{\partial t} + \mathbf{v}_0\cdot\nabla\langle s(t,\mathbf{x})\rangle = \nabla\cdot D(\mathbf{v}_0 t)\cdot\nabla\langle s(t,\mathbf{x})\rangle + O(\delta v^4) \tag{1.4}$$

where $D(\mathbf{v}_0 t) = \int_0^t \langle\delta\mathbf{v}(\mathbf{v}_0 t)\,\delta\mathbf{v}(\mathbf{v}_0 t')\rangle\,dt'$ is a diffusion matrix, $\mathbf{v}_0=\langle\mathbf{v}\rangle$ is the ensemble-averaged velocity, and $\delta\mathbf{v}=\mathbf{v}-\langle\mathbf{v}\rangle$ is a fluctuation about the average due to the rock heterogeneity and is a function of the spatial variables only for a stationary velocity field. The shape of the tracer interface at $t=0$ is given by $s(0,\mathbf{x})$. Equation (1.4) can also be derived from Taylor's Lagrangian theory of diffusion.[6] In the weak-heterogeneity limit, as we consider here, the term $O(\delta v^4)$ is negligible. Neuman and Zhang[27] developed a quasilinear theory for the growth of the mixing layer based on Corrsin's hypothesis. For strong heterogeneity, see refs. 10–12 for numerical studies. Resummation of perturbation series and high-order expansions are discussed in refs. 22 and 25.

In Section 2 we obtain an analytic expression for the growth of the mixing length, and show that the mixing layer is asymptotically anomalous when the correlation function decays slowly at large length scales. In Section 3, we study anomalous diffusion on finite length scales and an instantaneous fractal approximation. In Section 4, we show that to leading order, the asymptotic exponent of the correlation function of the permeability field determines whether the diffusion is asymptotically anomalous.

2. ASYMPTOTICALLY NORMAL AND ANOMALOUS DIFFUSION

In this section, we determine the anomalous growth of the mixing layer induced by a random velocity field. For simplicity, we consider the initial mixing interface to be a hyperplane with its normal direction parallel to $\mathbf{v}_0$. We choose $\mathbf{v}_0$ as the direction associated with the variable x_1. Let $\boldsymbol{\eta} = (\eta, x_2,..., x_d) = (x_1 - v_0 t, x_2,..., x_d)$. Here η is the first component of $\boldsymbol{\eta}$ and v_0 is the magnitude of $\mathbf{v}_0$. The untagged (tagged) fluid is located initially in the region $\eta > 0$ $(\eta < 0)$. Then in the weak-heterogeneity limit, Eq. (1.4) becomes

$$\begin{aligned} \langle s(t, \eta)\rangle_t &= \int_0^t q(v_0 \xi)\, d\xi\, \langle s(t, \boldsymbol{\eta})\rangle_{\eta\eta} = \alpha(v_0 t)\langle s(t, \boldsymbol{\eta})\rangle_{\eta\eta} \\ \langle s(0, \boldsymbol{\eta})\rangle &= s(0, \boldsymbol{\eta}) = \theta(-\eta) \end{aligned} \tag{2.1}$$

Here q is the velocity correlation function in the direction of x_1, θ is the Heaviside function, and

$$\alpha(v_0 t) = \int_0^t q(v_0 \xi)\, d\xi \tag{2.2}$$

is the diffusion coefficient in the x_1 direction, i.e., the first diagonal element of a diffusion matrix. Equation (2.1) is the convection-diffusion equation which governs the growth of the mixing layer.

The solution of Eq. (2.1) is

$$\langle s(t, \boldsymbol{\eta})\rangle = \frac{1}{2} \operatorname{erfc}\left(\frac{x_1 - v_0 t}{2[w(t)]^{1/2}}\right) \tag{2.3}$$

where erfc$(\cdot)$ is the complimentary error function and

$$w(t) = \int_0^t \int_0^\zeta q(v_0 \xi)\, d\zeta\, d\xi = \int_0^t (t - \xi)\, q(v_0 \xi)\, d\xi \tag{2.4}$$

Here $v_0 \xi$ represents the distance which the mixing interface travels over the time period ξ. We will suppress the coefficient v_0 in the argument of q for simplicity in the remaining formulas in this section and in the formulas of next section. In other words, we choose the units in such way that $v_0 = 1$.

The size of the mixing layer corresponds to the distance over which $\langle s\rangle$ varies from a value close to 0 to a value close to 1. Equation (2.3) shows that the mixing length $l(t)$ has the scaling

$$l(t) = 2[w(t)]^{1/2} = 2\left[\int_0^t (t - \xi)\, q(\xi)\, d\xi\right]^{1/2} \tag{2.5}$$

We assume that the correlation function q is a positive, piecewise continuous function. Then from Eq. (2.5) l is a positive, continuous, and piecewise differentiable function.

Any positive, piecewise differentiable function $f(t)$ can be expressed in a multifractal form, $f(t)=a(t)t^{b(t)}$, where $b(t)=d\ln f(t)/d\ln t$ and $\ln a(t)=\ln f(t)-\ln t\, d\ln f(t)/d\ln t$. In the $\ln[l(t)]-\ln(t)$ plane, $b(t)$ is the slope at $\ln t$. Therefore, the mixing length $l(t)$ can be expressed as

$$l(t)=c(t)t^{\gamma(t)} \tag{2.6}$$

Here $\gamma(t)$ is the exponent of the mixing length $l(t)$. From Eqs. (2.5) and (2.6) and the definition of Eq. (1.1), $\gamma(t)$, the exponent of the mixing length and $c(t)$ are given by

$$\gamma(t)=\frac{d\ln l(t)}{d\ln t}=\frac{t}{l}\frac{dl(t)}{dt}=\frac{1}{2}\left[1-\frac{\int_0^t \xi q(\xi)\,d\xi}{t\int_0^t q(\xi)\,d\xi}\right]^{-1} \tag{2.7}$$

and

$$\ln c(t)=\ln(2)+\frac{1}{2}\ln\left[\int_0^t (t-\xi)\,q(\xi)\,d\xi\right]-\frac{1}{2}\left[1-\frac{\int_0^t \xi q(\xi)\,d\xi}{t\int_0^t q(\xi)\,d\xi}\right]^{-1}\ln t$$

Equation (2.7) shows that the exponent $\gamma(t)$ of the mixing length at length scale t depends on all length scales less than t and that the correlation function $q(t)$ serves as a weight in the averaging process. This explains the dependence of the mixing length exponent on the flow history.

When q is differentiable, the exponent of q, according to Eq. (1.1), is given by

$$\beta(t)=\frac{d\ln q(t)}{d\ln t} \tag{2.8}$$

Let $\lambda(t)=\ln q(t)/\ln t$, $\beta_\infty=\lim_{t\to\infty}\beta(t)$ and $\lambda_\infty=\lim_{t\to\infty}\lambda(t)$.

Lemma 2.1. If q is differentiable at large length scales and if λ_∞ exists, then β_∞ exists and $\beta_\infty=\lambda_\infty$.

Proof. We first consider the case $\lambda_\infty\neq 0$:

$$\lambda_\infty=\lim_{t\to\infty}\lambda(t)=\lim_{t\to\infty}\frac{\lambda(t)\ln t}{\ln t}=\lim_{t\to\infty}\frac{d(\lambda(t)\ln t)}{d(\ln t)}=\lim_{t\to\infty}\frac{d(\ln q(t))}{d(\ln t)}=\beta_\infty$$

Here we have used l'Hôspital's rule, since both $\lambda(t)\ln t$ and $\ln t$ diverge as $t\to\infty$.

For $\lambda_\infty=0$, we define $\tilde{\lambda}(t)=\lambda(t)+1$; then $\lim_{t\to\infty}\tilde{\lambda}(t)=1$. Replacing λ by $\tilde{\lambda}$ in the above proof for $\lambda_\infty\neq 0$, we will have $\beta_\infty=0$.

Proposition 2.2. If β_∞ exists, then γ_∞ exists. The asymptotic diffusion is normal when $\beta_\infty < -1$ and anomalous when $-1 \leqslant \beta_\infty$. The asymptotic exponent of the mixing length is given by

$$\gamma_\infty = \max\left\{\frac{1}{2}, 1 + \frac{\beta_\infty}{2}\right\} \tag{2.9}$$

Proof. Since

$$\lim_{t \to \infty} \frac{\int_0^t \xi q(\xi)\, d\xi}{t \int_0^t q(\xi)\, d\xi} = 0 \qquad \text{for} \quad \lambda_\infty < -1$$

(which is equivalent to $\beta_\infty < -1$ from Lemma 2.1), we have

$$\gamma_\infty = \tfrac{1}{2}, \qquad \text{for} \quad \beta_\infty < -1 \tag{2.10}$$

This is normal diffusion.

When $-1 \leqslant \lambda_\infty$ (i.e., $-1 \leqslant \beta_\infty$), both $\int_0^t \xi q(\xi)\, d\xi$ and $t \int_0^t q(\xi)\, d\xi$ diverge as $t \to \infty$. We apply l'Hôspital's rule to obtain the limit

$$\lim_{t \to \infty} \frac{\int_0^t \xi q(\xi)\, d\xi}{t \int_0^t q(\xi)\, d\xi} = \lim_{t \to \infty} \frac{1}{1 + 1/(1 + \beta(t))} = \frac{1 + \beta_\infty}{2 + \beta_\infty} \qquad \text{when} \quad -1 \leqslant \beta_\infty$$

Here we have used Eq. (2.8). Therefore we obtain the result

$$\gamma_\infty = 1 + \frac{\beta_\infty}{2} \qquad \text{for} \quad -1 \leqslant \beta_\infty \tag{2.11}$$

The diffusion is anomalous in this case.

Combining (2.10) and (2.11), Eq. (2.9) follows.

Notice that, from Eq. (2.11), $\gamma_\infty = 1/2$ for $\beta_\infty = -1$. Therefore γ_∞ is a continuous function of β_∞. Here $\beta_\infty = -1$ is the critical point, joining the regime of normal diffusion to the regime of anomalous diffusion. At the critical point, $\int_0^t \xi q(\xi)\, d\xi$ diverges linearly, and $\int_0^t q(\xi)\, d\xi$ diverges logarithmically. Therefore $\gamma(t)$ approaches 1/2 at a rate $(\ln t)^{-1}$. The convergence to the asymptotic limit is the slowest at the critical point.

The condition $\beta_\infty < -1$ is equivalent to the condition that the diffusion coefficient $\alpha(t)$ converges at large length scales. In this case, α approaches a constant at large length scales. If $\beta_\infty > -1$, α diverges at large length scales. The diffusion is asymptotically anomalous at the critical point $\beta_\infty = -1$, although $\lim_{t \to \infty} \gamma(t) = 1/2$, since $\alpha(t)$ diverges as $\ln t$. Therefore we have the following criterion:

Corollary 2.3. The diffusion is asymptotically Fickian if and only if the limit $\lim_{t \to \infty} \alpha(t)$ converges. Otherwise it is anomalous.

From the positivity of q, Eq. (2.2) implies that the diffusion coefficient $\alpha(t)$ is a nondecreasing function of the length scale. It explains the observation from experimental and field data that the dispersivity is an increasing function of length scale. In Fickian diffusion, the diffusion coefficient is a constant over all length scales. Therefore the correlation function q is proportional to a delta function at origin. This corresponds to a white noise random field $\delta \mathbf{v}$.

3. MULTIFRACTAL ANOMALOUS DIFFUSION

Here we determine the behavior of the mixing layer on finite length scales from the properties of the velocity correlation function. We study the transient effect on different length scales and find the conditions which allow $\gamma(t)$ to be determined from $\beta(t)$, the exponent of the velocity correlation function $q(\xi)$.

Proposition 3.1.

The following properties of γ can be determined from the properties of q:

(a) $1/2 \leqslant \gamma(t)$ for $0 \leqslant t < \infty$.

(b) If q is a fractal field, $q(\xi) = c\xi^{\beta}$, then $\gamma(t) = \max\{1/2, 1 + \beta/2\}$ for $0 \leqslant t < \infty$.

(c) If $q(\xi)$ is nonsingular at short length scales and $q(0) \neq 0$, then $\gamma(0) = 1$. If $q(\xi)$ diverges as $c\xi^{\beta}$, for $\xi \ll 1$, with $\beta < 0$, then $\gamma(0) = \max\{1/2, 1 + \beta/2\}$.

(d) If $q(\xi)$ is a nonincreasing (nondecreasing) function and $\lim_{\xi \to 0} \xi q(\xi) = 0$, then $\gamma(t) \leqslant 1$ ($\geqslant 1$) for $0 \leqslant t < \infty$.

(e) Let $z(t) = 2\gamma(t) - tq(t)/\alpha(t)$. Then $\gamma(t)$ will increase (decrease) when $z(t) < 1$ (> 1). $\gamma(t)$ is a stationary value when $z(t) = 1$.

Proof. (a) follows from the positivity of $q(\xi)$. (b) and (c) follow from Eq. (2.7) with the correspondingly stated properties of $q(\xi)$.

To prove (d), we consider the integral

$$\int_0^t \left(\frac{1}{2} t - \xi\right) q(\xi)\, d\xi = \frac{1}{2} (t - \xi)\, \xi q(\xi) \big|_0^t - \frac{1}{2} \int_0^t (t - \xi)\, \xi q'(\xi)\, d\xi \geqslant 0$$

for the case in which $q(\xi)$ is a nonincreasing function. The expression on the right-hand side of the equal sign is obtained by integration by parts.

This inequality is equivalent to $\int_0^t \xi q(\xi)\, d\xi / t \int_0^t q(\xi)\, d\xi \leqslant 1/2$. From Eq. (2.7), we have $\gamma(t) \leqslant 1$. The proof for the case of nondecreasing $q(\xi)$ is similar.

By taking the derivative of Eq. (2.7) with respect to t and using the definitions given by Eqs. (2.2) and (2.7), the results of (e) follow. Therefore Proposition 3.1 holds.

If the exponent β of q varies slowly over some length scale t, we may expect the mixing growth rate $\gamma(t)$ to approach the behavior of a fractal field, if the fluid has been travelling on that length scale long enough. Furthermore, we may also expect that the slowly varying condition is not necessary for $\beta(t) < -1$, since for all $\beta(t) < -1$, the asymptotic limit of $\gamma(t)$ is the same i.e. $\frac{1}{2}$. In fact when $q(\xi)$ has rapid decay on certain length scales, the diffusion coefficient is approximately constant (increasing slowly) on those scales. Therefore, when either q decays rapidly or β varies slowly, we may expect that the exponent of the mixing length approaches an asymptotic (fractal) limit: $\gamma_{\text{fr}}(t) = \max\{1/2, 1 + \beta(t)/2\}$. We study such an instantaneous fractal field approximation and the transient effects below.

We consider the case of rapidly decaying q first. For convenience, we introduce

$$I_n(t) \equiv \int_0^t \xi^n q(\xi)\, d\xi \tag{3.1}$$

Lemma 3.2. Let $\gamma_{\max} = \max\{\gamma(t): t_1 \leqslant t \leqslant t_2\}$. Then

$$\gamma(t) \leqslant \frac{1}{2} + \gamma_{\max} \frac{I_1(t)}{t I_0(t)} \qquad \text{for} \quad t_1 \leqslant t \leqslant t_2 \tag{3.2}$$

Proof. Let $R = I_1(t)/t I_0(t)$. From Eq. (2.7) we have

$$\begin{aligned} \gamma(t) &= \frac{1}{2(1-R)} = \frac{1}{2} + \frac{R}{2(1-R)} \\ &\leqslant \frac{1}{2} + \frac{R}{2[1-\max(R)]} = \frac{1}{2} + \gamma_{\max} R \qquad \text{for} \quad t_1 \leqslant t \leqslant t_2 \end{aligned}$$

Proposition 3.3. If there exist positive constants c and δ such that $q(\xi) \leqslant c \xi^{-1-\delta}$ for $t_1 \leqslant \xi \leqslant t_2$, then there exist two constants c_1 and c_2 such that for $\delta \neq 1$, and $t_1 \leqslant t \leqslant t_2$,

$$\gamma(t) \leqslant \tfrac{1}{2} + c_1 t^{-1} + c_2 t^{-\delta} \tag{3.3}$$

and for $\delta = 1$, and $t_1 \leqslant t \leqslant t_2$,

$$\gamma(t) \leqslant \tfrac{1}{2} + c_1 t^{-1} + c_2 t^{-1} \ln t \tag{3.4}$$

Proof. Consider the case $\delta \neq 1$ first. From Lemma 3.2 we have

$$\begin{aligned}
\gamma(t) &\leqslant \frac{1}{2} + \gamma_{\max} \frac{I_1(t_1) + \int_{t_1}^{t} \xi q(\xi)\, d\xi}{t[I_0(t_1) + \int_{t_1}^{t} q(\xi)\, d\xi]} \\
&\leqslant \frac{1}{2} + \frac{\gamma_{\max}}{tI_0(t_1)} \left[I_1(t_1) + \frac{c}{1-\delta}(t^{1-\delta} - t_1^{1-\delta})\right] \\
&= \frac{1}{2} + \gamma_{\max} t_1 \left[\frac{2\gamma(t_1) - 1}{2\gamma(t_1)} - \frac{c t_1^{-\delta}}{(1-\delta)\, I_0(t_1)}\right] t^{-1} \\
&\quad + \gamma_{\max} \frac{c}{(1-\delta)\, I_0(t_1)} t^{-\delta} \qquad \text{for} \quad \delta \neq 1
\end{aligned}$$

Therefore

$$c_1 = \gamma_{\max}\, t_1 \left[\frac{2\gamma(t_1) - 1}{2\gamma(t_1)} - \frac{c t_1^{-\delta}}{(1-\delta)\, I_0(t_1)}\right] \quad \text{and} \quad c_2 = \gamma_{\max} \frac{c}{(1-\delta)\, I_0(t_1)}$$

Following the same procedure with $\delta = 1$, we have

$$c_1 = \gamma_{\max}\, t_1 \left[\frac{2\gamma(t_1) - 1}{2\gamma(t_1)} - \frac{c \ln t_1}{I_0(t_1)}\right] \quad \text{and} \quad c_2 = \gamma_{\max} \frac{c}{I_0(t_1)}$$

For the system to approach a normal diffusion limit, two conditions must be satisfied: (1) the decay of the correlation function must be faster than $1/t$; (2) the transient effect induced from the history $0 \leqslant \xi \leqslant t_1$ must be damped. The term $c_1 t^{-1}$ corresponds to diminishing the transient effect and the term $c_2 t^{-\delta}$ represents the rate at which the system will approach the limit $\gamma = 1/2$ if the transient effect is negligible. When $\delta \leqslant 1$ the convergence rate of γ is dominated by the decay rate of the correlation function. When $\delta > 1$ the convergence rate of γ is dominated by the damping of the transient effect.

We study the exponent of the mixing length for slowly varying β below. The approximation of γ in this case depends on an approximate evaluation of the integrals $I_0(t)$ and $I_1(t)$.

Lemma 3.4. Let b be the maximum variation of the exponent of q over the range $[t_1, t_2]$, $b = \max\{|\beta(\xi) - \beta(\xi')| : t_1 \leqslant \xi, \xi' \leqslant t_2\}$. Let $\tilde{I}_n(t)$ denote the approximate value of $I_n(t)$ from a fractal approximation to q in the portion of the integration over the interval $[t_1, t]$. Then, in the interval $[t_1, t_2]$, the relative error in $\tilde{I}_n$ is bounded by

$$\left|\frac{I_n(t)-\tilde{I}_n(t)}{I_n(t)}\right| \leqslant \frac{b}{|n+1+\beta(t)|} \qquad \text{for} \quad \beta(t) \neq -(n+1) \tag{3.5}$$

with

$$\tilde{I}_n(t) = I_n(t_1) + \frac{1}{n+1+\beta(t)} [t^{n+1}q(t) - t_1^{n+1}q(t_1)]$$

and

$$\left|\frac{I_n(t)-\tilde{I}_n(t)}{I_n(t)}\right| \leqslant b(|\ln t| + |\ln t_1|) \qquad \text{for} \quad \beta(t) = -(n+1) \tag{3.6}$$

with

$$\tilde{I}_n(t) = I_n(t_1) + t^{n+1}q(t)\ln t - t_1^{n+1}q(t_1)\ln t_1$$

Proof of (3.5).

$$I_n(t) = I_n(t_1) + \int_{t_1}^{t} \xi^n q(\xi)\, d\xi = I_n(t_1) + \int_{t_1}^{t} \xi^{n+\beta(t)}\xi^{-\beta(t)} q(\xi)\, d\xi$$

Integration by parts on the portion $\xi^{n+\beta(t)}$ of the integrand yields

$$I_n(t) = \tilde{I}_n(t) + \frac{1}{n+1+\beta(t)} \int_{t_1}^{t} [\beta(t) - \beta(\xi)]\, \xi^n q(\xi)\, d\xi$$

Therefore we have

$$|I_n(t) - \tilde{I}_n(t)| \leqslant \frac{b}{|n+1+\beta(t)|} \int_{t_1}^{t} \xi^n q(\xi)\, d\xi \tag{3.7}$$

Dividing both sides of Eq. (3.7) by $I_n(t)$, (3.5) follows.

The proof of (3.6) is similar.

Let $e_n(t) = [I_n(t) - \tilde{I}_n(t)]/I_n(t)$ for $n = 0, 1$ be the relative error in a fractal approximation of the integrals I_0 and I_1. From Lemma 3.4, $\tilde{I}_n(t)$ can be expressed as $\tilde{I}_n(t) = I_n(t_1) + F_n(t) - F_n(t_1)$, where

$$F_n(\xi) = \xi^{n+1}q(\xi)/[n+1+\beta(\xi)] \qquad \text{for} \quad \beta(t) \neq -(n+1)$$

or

$$F_n(\xi) = \xi^{n+1}q(\xi)\ln\xi \qquad \text{for} \quad \beta(t) = -(n+1)$$

Let $r_n(t)$ be the ratio $r_n(t) = [I_n(t_1) - F_n(t_1)]/F_n(t)$ for $n = 0, 1$. The relative error from the fractal approximation at length scale t is represented by $e_n(t)$, and $r_n(t)$ represents the remaining transient effect at length scale t.

Proposition 3.5. If $q(t)$ is differentiable on $[t_1, t_2]$ and satisfies the conditions

(a) $\max\{1, |1 + \beta(t)|\} |e_n(t)| \ll 1$, with $-1 \leqslant \beta(t)$; and

(b) there exists t', $t_1 \leqslant t' < t_2$, such that $|r_n(t)| \ll 1$ whenever $t' \leqslant t \leqslant t_2$;

then $\gamma(t) = 1 + \frac{1}{2}\beta(t) + O(e(t) + r(t))$ for $t' \leqslant t \leqslant t_2$.

Proof. Substituting the expressions of $\tilde{I}_n(t)$ in Lemma 3.4 into Eq. (2.7) and applying Taylor's expansion with the conditions stated in (a) and (b), the result follows. The leading-order contribution from $O(e(t) + r(t))$ is given by

$$[2 + \beta(t)][1 + \beta(t)][e_1(t) - e_0(t) + r_1(t) - r_0(t)]/2$$

The condition (a) represents the fact that the variation of $\beta(t)$ has to be slow enough to let fluids realize that the correlation function is approximately fractal at that length scale. Condition (b) represents the fact that the slowly varying condition must persist sufficiently long to diminish the transient effect. After time t', the transient effects are negligible. Since the asymptotic limit of $r_n(t)$ is zero, condition (b) can be satisfied when the range of slow variation is sufficiently large.

Combining Propositions 3.3 and 3.5, we have following corollary.

Corollary 3.6. If $q(t)$ satisfies the rapid decay condition stated in Proposition 3.3 or the slowly varying condition stated in Proposition 3.5, then

$$\gamma(t) \approx \gamma_{\mathrm{fr}}(t) = \max\left\{\frac{1}{2}, 1 + \frac{\beta(t)}{2}\right\} \tag{3.8}$$

The experimental data and field data[13] indeed show that the diffusion is typically anomalous in the range from 10 cm to 1 km. The data[13] on length scales larger than 1 km are too scattered to determine its asymptotic behavior.

To illustrate anomalous difusion in a multifractal field, we consider a simple velocity correlation function

$$q(x) = q_0(1 + x/a)^{\beta} \tag{3.9}$$

Here $q_0 = q(0)$ is the variance of the velocity field, a is a characteristic length, and β is an asymptotic exponent. This model allows a simple interpolation between the distinct behavior of the fluids at short and long length scales. By introducing a dimensionless length $\tau = x/a$, one can express the exponent of the mixing layer as

$$\gamma(\tau) = \begin{cases} \dfrac{1}{2}\dfrac{\tau(2+\beta)[(1+\tau)^{1+\beta}-1]}{(1+\tau)^{2+\beta}-(2+\beta)\tau-1} & \text{for} \quad \beta \neq -1, -2 \\ \dfrac{1}{2}\dfrac{\tau\ln(1+\tau)}{(1+\tau)\ln(1+\tau)-\tau} & \text{for} \quad \beta = -1 \\ \dfrac{1}{2}\dfrac{\tau^2}{(1+\tau)[\tau-\ln(1+\tau)]} & \text{for} \quad \beta = -2 \end{cases} \tag{3.10}$$

The diffusion coefficient can be expressed as

$$\alpha(\tau) = \begin{cases} \dfrac{q_0}{v_0}\dfrac{1}{1+\beta}[(1+\tau)^{1+\beta}-1] & \text{for} \quad \beta \neq -1 \\ \dfrac{q_0}{v_0}\ln(1+\tau) & \text{for} \quad \beta = -1 \end{cases} \tag{3.11}$$

The exponent of q is given by $\beta(\tau) = \beta\tau/(1+\tau)$. Then

$$\gamma_{\mathrm{fr}}(\tau) = \max\left\{\frac{1}{2}, 1 + \frac{\beta}{2}\frac{\tau}{1+\tau}\right\}$$

From Proposition 3.1, $\gamma(t)$ has following properties: $\gamma(0) = 1$, $\gamma(t) \leqslant 1$ for $\beta \leqslant 0$. Since $\beta_\infty = \beta$, we have $\gamma_\infty = \max\{1/2, 1+\beta/2\}$. In Fig. 1 we show γ as a function of $-\beta$ at different times, i.e., at different length scales. Figure 1 shows that γ converges to $1+\beta/2$ for $-1 \leqslant \beta$ and to $1/2$ for $\beta < -1$. At the critical point $\beta = -1$, γ has the slowest rate of approach to its asymptotic limit. In Fig. 2, we compare the exact value $\gamma(\tau)$ with the value of the instantaneous fractal exponent

$$\gamma_{\mathrm{fr}}(\tau) = \max\left\{\frac{1}{2}, 1 + \frac{\beta(\tau)}{2}\right\}$$

$\gamma(\tau)$ is shown as a solid curve and $\gamma_{\mathrm{fr}}(\tau)$ as a dashed curve in Fig. 2. It shows that $\gamma_{\mathrm{fr}}(\tau)$ agrees with $\gamma(\tau)$ very well over all length scales for $-0.5 < \beta \leqslant 0$. The smaller β is, the slower the variation of $\beta(\tau)$ is, and the better $\gamma_{\mathrm{fr}}(\tau)$ agrees with $\gamma(\tau)$. For $\beta \leqslant -0.5$, $\gamma_{\mathrm{fr}}(\tau)$ deviates from $\gamma(\tau)$ on the intermediate length scales, $\ln\tau \approx 1$, while it still agrees with $\gamma(\tau)$ very well at short and large length scales. At short and large length scales, the correlation function, Eq. (3.9), is approximately fractal. The transient effect

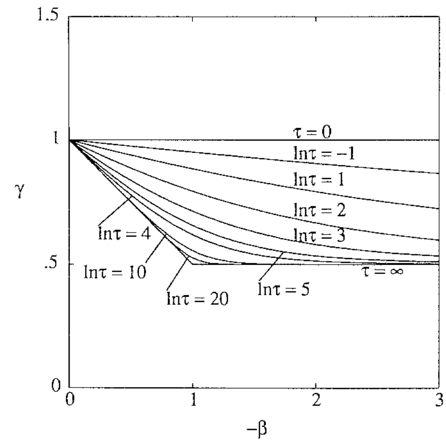

Fig. 1. Plot of the exponent $\gamma(\tau)$ of the mixing length vs. the asymptotic velocity exponent β at different times (length scales). Here the correlation function of the velocity field has a simple form $q(\tau) = q_0(1+\tau)^{\beta}$. In agreement with the general theory, $\gamma(0) = 1$ for all values of β and $\gamma_\infty = \max\{1/2, 1 + \beta/2\}$. The diffusion is anomalous ($\gamma > 1/2$) at any finite length scales. The asymptotic diffusion is anomalous when $-1 \leqslant \beta$, and normal when $\beta < -1$. The critical point $\beta = -1$ has the slowest rate of approaching the asymptotic limit.

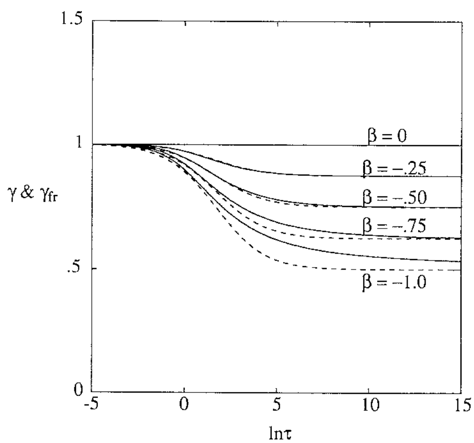

Fig. 2. Comparison of the exact exponent $\gamma(\tau)$ and instantaneous fractal exponent $\gamma_{fr}(\tau)$, respectively, for the mixing length growth plotted vs. $\ln\tau$. Here $\gamma(\tau)$ is shown as the solid curves and $\gamma_{fr}(\tau)$ as the dashed curves. The difference between the solid curves and the dashed curves indicates the variation of the $\beta(\tau)$ and the transient effect between short and large length scales.

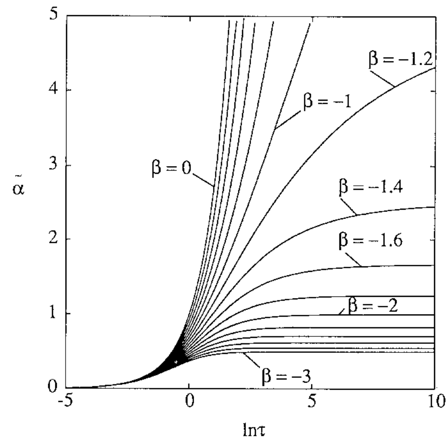

Fig. 3. The dimensionless diffusion coefficient as a function of time for different values of β. The values of β are evenly spaced from $\beta = -3$ to $\beta = 0$ with an increment of 0.2. By comparing with Fig. 1, one can see that the asymptotic diffusion is anomalous (normal) when the diffusion coefficient diverges (converges).

at short length scales is negligible and the transient effect is diminished at sufficiently large length scales. In Fig. 3 we plot the dimensionless diffusion coefficient $\tilde{\alpha}(\tau) = \alpha(\tau) v_0/q_0$ as a function of τ for different values of β. This figure should be compared with Fig. 1. It shows that the diffusion coefficient diverges for asymptotically anomalous diffusion $(-1 \leqslant \beta)$ and converges for asymptotically normal diffusion $(\beta < -1)$.

4. ROCK HETEROGENEITY

In this section we determine the asymptotic exponent of the mixing layer from the asymptotic exponent of the correlation function of rock permeability. Let $g(x - x') = \langle \varepsilon_1(x)\, \varepsilon_1(x') \rangle$ be the correlation function of the permeability field. Let $\rho(x) = d \ln g(x)/d \ln x$ be the exponent of $g(x)$, and $\rho_\infty = \lim_{x \to \infty} \rho(x)$. Then $\gamma_\infty = \max\{1/2, 1 + \rho_\infty/2\}$ for dimension greater or equal to 2.

The permeability tensor is diagonalizable. Without losing generality, we choose our coordinate system to coincide with the principal axis. We express the permeability field as $K = K_0 e^{\varepsilon(\mathbf{x})}$. Here K_0 is a constant tensor normalized so that ε has mean zero and $\varepsilon(\mathbf{x})$ is a Gaussian random field. The ith entries in K_0 and ε are k_i and ε_i, respectively, which determine the permeability along the principal axis x_i. We assume that the tracer flow is moving along one of the principal axes x_1.

Let G be the Green's function determined by the equation

$\nabla \cdot (K_0 \nabla G(\mathbf{x}, \mathbf{x}')) = k_1\, \delta(\mathbf{x} - \mathbf{x}')$, where $\delta(\mathbf{x} - \mathbf{x}')$ is a delta function in d dimensions. Let $G' = \partial G/\partial x_1$, $g' = dg/d(v_0 t)$, and $g'' = d^2 g/d(v_0 t)^2$. Let $*$ denote convolution.

Lemma 4.1. When $\|\varepsilon\| \ll 1$, the velocity field correlation function q is given by

$$q(\xi) = v_0^2[g(\xi) - 2(G' * g')(\xi) + (G' * G' * g'')(\xi)] + O(\varepsilon^4) \qquad (4.1)$$

where $\xi = v_0(t - t')$.

Proof. By linearizing Eqs. (1.3) in terms of ε and evaluating the velocity correlation function in the direction of x_1, the result follows.

Proposition 4.2. To leading order in ε, the asymptotic exponent of the mixing length is given by $\gamma_\infty = \max\{1/2, 1 + \rho_\infty/2\}$ for dimension $\geqslant 2$, where ρ_∞ is the asymptotic exponent of g.

Proof. In Eq. (4.1), the asymptotic decay rate of $G' * G' * g''$ is at least as fast as that of $G' * g'$. Therefore the asymptotic decay of Eq. (4.1) is determined by the slower of g and $G' * g'$. The asymptotic decay of $G' * g'$ is at least as fast as $|G'| * |g'|$, which is determined by the slower one between $|G'|$ and $|g'|$. Now, $|g'|$ decays faster than g and $|G'|$ decays at least as fast as $|\mathbf{x}|^{-1}$ for dimension $\geqslant 2$. The only possibility for $-1 < \beta_\infty$ is $-1 < \rho_\infty$, Therefore $\beta_\infty = \rho_\infty$ when $-1 < \rho_\infty$. The proposition holds.

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